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REFLECTION AND TRANSMISSION OF WAVES FROM AN INTERFACE WITH A PHASE-TRANSFORMING SOLID

by

Rohan Abeyaratne[1] and James K. Knowles[2]

[1] Department of Mechanical Engineering
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

[2] Division of Engineering and Applied Science
California Institute of Technology
Pasadena, California 91125

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ABSTRACT

This paper is concerned with waves in a composite elastic bar, the left half of which is composed of a linearly elastic material, while the nonlinearly elastic material of the right half can undergo a phase transition. We assume that a wave in the left portion of the bar is incident upon the interface between the two materials, and we investigate the question of whether the phase transition can be exploited to augment or diminish the strength of the reflected or transmitted wave.

1. Introduction. We consider a composite tensile specimen consisting of two dissimilar elastic bars joined end-to-end. One of the two elastic materials ("material 2") is capable of undergoing a stress-induced transition to a second phase, while the other ("material 1") is not. When an incident wave traveling in the single-phase material strikes the bimaterial interface, it may nucleate a phase transition in material 2. The reflection and transmission characteristics of the product phase in such a transition will in general differ from those of the parent phase. We designate as "material 3" the *single*-phase elastic material whose properties are identical with those of the parent phase of material 2. Our interest lies in comparing the strengths, relative to the incident wave, of the reflected and transmitted waves generated in the material 1/material 2 composite bar with the relative strengths of those that would occur in a bar composed of material 1 and material 3. In particular, we investigate the extent to which the phase-transforming capability of material 2 can be exploited to control the strength of the reflected or transmitted wave.

The composite bar is viewed as consisting of two perfectly bonded, semi-infinite one-dimensional elastic continua. Material 1 is taken to be linearly elastic, while material 2 is treated on the basis of a nonlinear continuum model of the kind currently receiving much attention in discussions of the macroscopic aspects of phase transformations of martensitic type; see, for example the references cited in [1]. This model is based on a "two-well" potential energy for material 2, and it includes a nucleation criterion and a kinetic relation governing the initiation and evolution, respectively, of the phase transition. Thermal effects are neglected here, so that the model is a purely mechanical one. It has been applied to quasi-static phase transformations in [2], where it was shown to lead to results in qualitative agreement with some experimental observations on materials of the shape-memory type. The dynamics of the model have been studied in [3].

There are many recent papers devoted to the general issue of the continuum-mechanical modeling of the macroscopic effects of phase transformations; examples may be found in the references cited in [1]. The only work of which we are aware that is related to the specific problem under discussion here is that of Pence [4, 5]. In [4], Pence studies the reflection and transmission of an acoustic shear wave from an initially stationary phase boundary in an elastic solid. The analysis in [5] is concerned with the structure of the fields in two elastic bars, one of which (the impactor) is composed of a single-phase material, while the other (the target) is made of a material capable of sustaining a phase transformation. Neither of these papers addresses the issues of primary interest here.

We describe the basic model to be used in the following section. Section 3 contains the formulation of the underlying wave propagation problem, and Section 4 is devoted to its solution. In Section 5, we show that the ratio of the relative strength of the reflected wave in the presence of the phase transition to its relative strength in the absence of the transition is governed by two material parameters: one is the ratio of the mechanical impedances of material 1 and the

parent phase of material 2, while the other parameter is inherently related to the phase-transition properties of material 2. In terms of these two parameters, we derive in Section 5 conditions under which (i) the reflectivity is always increased by the phase transition, independently of the kinetic relation and the nucleation criterion, (ii) the reflectivity is always decreased, and (iii) the effect of the transition on reflectivity depends on the details of kinetics and nucleation. Some analogous results for the transmitted wave are stated without proof in Section 6.

2. The model. In a reference configuration, the composite bar occupies the entire x-axis, with material 1 in $x < 0$, material 2 in $x > 0$; the referential cross-sectional area is A. We treat longitudinal motions in which a particle at x in the reference state is carried to the point $x + u(x, t)$ at time t. The displacement u is to be continuous with piecewise continuous first and second derivatives. The strain and particle velocity are defined by $\gamma(x, t) = u_x(x, t)$ and $v(x, t) = u_t(x, t)$, respectively, where the derivatives exist; in order to assure that the mapping $x \to x + u$ is one-to-one, we require that $\gamma > -1$ everywhere. At points in the x,t-plane where the fields are smooth, balance of linear momentum in the absence of body force requires that

$$\sigma_x = \rho(x)\, v_t\,, \tag{2.1}$$

where $\sigma(x, t)$ is the stress in the bar, and

$$\rho(x) = \begin{cases} \rho_1\,, & x < 0, \\ \rho_2\,, & x > 0; \end{cases} \tag{2.2}$$

the constants ρ_1 and ρ_2 are the mass densities of materials 1 and 2 in the reference state, respectively. From the definitions of γ and v, one also has

$$v_x - \gamma_t = 0 \tag{2.3}$$

where the fields are smooth.

The stress-strain relation for the composite material 1/material 2 bar is

$$\sigma = \hat{\sigma}(\gamma, x) \equiv \begin{cases} \hat{\sigma}_1(\gamma), & x < 0, \quad \gamma > -1, \\ \hat{\sigma}_2(\gamma), & x > 0, \quad \gamma > -1, \end{cases} \tag{2.4}$$

where $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are the respective stress response functions of materials 1 and 2 . For material 1, we assume that

$$\hat{\sigma}_1(\gamma) = \mu_1 \gamma, \quad -1 < \gamma < \infty, \tag{2.5}$$

where μ_1 is Young's modulus. The stress response function for material 2 is taken to have the following "trilinear" form:

$$\hat{\sigma}_2(\gamma) = \begin{cases} \mu_2 \gamma, & -1 < \gamma \le \gamma_M, \\ \sigma_M - (\sigma_M - \sigma_m)(\gamma - \gamma_M)/(\gamma_m - \gamma_M), & \gamma_M \le \gamma \le \gamma_m, \\ \mu_2 (\gamma - \gamma_0), & \gamma \ge \gamma_m. \end{cases} \tag{2.6}$$

The graph of $\hat{\sigma}_2(\gamma)$ is shown schematically in Figure 1; the significance of the constants μ_2, γ_0, γ_M, γ_m, σ_M and σ_m may be read from the graph. Each of the three branches of the stress-strain curve in the figure is identified with a *phase* of material 2: the two rising branches are the low-strain and high-strain phases, and the declining branch is the unstable phase. For simplicity, we have assumed that the Young's modulus μ_2 is the same in both the low-strain and high-strain

phases of material 2, but this is by no means necessary. The *transformation strain* γ_0 is the distance between the low-strain and high-strain branches of the stress-strain curve at any given value of stress; because $\sigma_M > \sigma_m$, one has $\gamma_0 > \gamma_m - \gamma_M$. If $\gamma_0 < \gamma_m$, the stress σ_m at the local minimum is positive, as in the figure. If $\gamma_0 \geq \gamma_m$, this minimum stress is non-positive, and the right half of the bar possesses more than one unstressed configuration. The stress σ_0 shown in Figure 1 is such that the shaded areas are equal; σ_0 is called the *Maxwell stress*. It is given by

$$\sigma_0 = \frac{1}{2}(\sigma_M + \sigma_m) = \frac{\mu_2}{2}(\gamma_M + \gamma_m - \gamma_0); \tag{2.7}$$

for simplicity, we assume that $\sigma_0 > 0$.

Equations (2.1)-(2.6) yield a system of two differential equations for γ and v for $x < 0$ corresponding to material 1, and a second pair of differential equations for γ and v in the interval $x > 0$ for the part of the bar made of material 2.

Suppose that there is a discontinuity in strain or particle velocity along the curve $x = s(t)$ in the x,t-plane. Balance of linear momentum and the assumed smoothness of the motion require that the following jump conditions hold:

$$[[\sigma]] = -\dot{s}\,[[\rho v]], \tag{2.8}$$

$$\dot{s}\,[[\gamma]] = -[[v]], \tag{2.9}$$

where for any $g(x, t)$, we write $[[g]] = [[g(x,t)]] \equiv g(s(t)+, t) - g(s(t)-, t)$.

The strain energy per unit reference volume for the composite bar is

$$W(\gamma, x) = \begin{cases} W_1(\gamma), & x < 0, \ \gamma > -1, \\ W_2(\gamma), & x > 0, \ \gamma > -1, \end{cases} \tag{2.10}$$

where the separate energy densities for materials 1 and 2 are given by

$$W_k(\gamma) = \int_0^{\gamma} \hat{\sigma}_k(\gamma')\, d\gamma', \quad \gamma > -1, \quad k = 1,2. \tag{2.11}$$

The *potential energies* for the two materials are defined by $G_k(\gamma, \sigma) = W_k(\gamma) - \sigma\gamma$. For every fixed σ, $G_1(\gamma, \sigma)$ as a function of γ has a single minimum at $\gamma = \sigma/\mu$. In contrast, the potential G_2 for material 2 is such that, at a fixed value of σ between σ_m and σ_M, $G_2(\cdot, \sigma)$ has two local minima separated by a local maximum. When σ is outside this range, $G_2(\cdot, \sigma)$ has only one minimum. Thus when the stress is such that material 2 can exist in either the low- or the high-strain phase, G_2 is a "two-well" potential typical of two-phase materials.

Suppose that in a piece of the bar corresponding to $x_1 \le x \le x_2$, the strain and particle velocity are discontinuous at $x = s(t)$ but are otherwise smooth. Writing

$$E(t) = \int_{x_1}^{x_2} \left[W(\gamma(x, t), x) + \frac{1}{2}\rho(x)\, v^2(x, t) \right] A\, dx \tag{2.12}$$

for the total energy in this portion of the bar at time t, one can show by a direct calculation that

$$\sigma(x_2, t)Av(x_2,t) - \sigma(x_1, t)Av(x_1, t) - \dot{E}(t) = f(t)A\, \dot{s}(t), \tag{2.13}$$

where

$$f(t) = [[W(\gamma(x,t), x)]] - <\sigma(x,t)> [[\gamma(x, t)]], \tag{2.14}$$

and $<\sigma(x, t)> = (1/2)\{\sigma(s(t)+,t) + \sigma(s(t)-,t)\}$. We call f(t) the *driving traction acting on the discontinuity* at x = s(t). In view of (2.13), one may think of $f(t)A\,\dot{s}(t)$ as the instantaneous dissipation rate associated with the discontinuity; if either f(t) = 0 or the discontinuity is stationary (in the Lagrangian sense) so that $\dot{s}(t) = 0$, this dissipation rate vanishes. At any discontinuity, it is required that the dissipation rate be non-negative:

$$f(t)\,\dot{s}(t) \geq 0\,. \tag{2.15}$$

One can show that, if the motion of the bar is viewed as occurring isothermally, (2.15) is a consequence of the second law of thermodynamics; we shall therefore refer to (2.15) as the *entropy inequality.*

In the problem to be treated here, we shall be concerned with three different types of strain discontinuity. The first of these occurs at the bimaterial interface at x = 0; because this discontinuity is stationary, (2.15) is trivially satisfied. The second type of strain discontinuity to be encountered involves a strain jump *either* in material 1 *or* between strains $\gamma(s(t)\pm,t)$, both of which belong to the *same* phase in material 2. Such a discontinuity is a *sound wave*. Because the stress-strain relation is linear between the strains $\gamma(s(t)-, t)$ and $\gamma(s(t)+, t)$ in either of these circumstances, the definition (2.14) yields f(t) = 0, so that (2.15) is automatically satisfied at a sound wave as well. Finally, we shall need to deal with strain jumps in material 2 for which $\gamma(s(t)-, t)$ is in the high-strain phase while $\gamma(s(t)+, t)$ is in the low-strain phase. Such a discontinuity is an example of a *phase boundary*; for the problem to be considered here, phase boundaries in material 2 will always move to the right, so that $\dot{s}(t) > 0$. It then follows from (2.15) that f(t) must be non-negative at the phase boundaries arising in the present problem. Moreover, it is easy to show with the help of (2.14), (2.10), (2.11) for k=2, (2.6) and (2.7) that, at

any phase boundary with high strain on the left, low strain on the right, the driving traction is given by

$$f(t) = \frac{\mu_2}{2}\gamma_0\left[\overset{+}{\gamma}(t) + \overset{-}{\gamma}(t) - \gamma_M - \gamma_m\right]. \tag{2.16}$$

As in the analyses in [2,3], a *kinetic relation* is to be prescribed at a phase boundary; we take it to have the form of a relation between driving traction f and phase boundary velocity $\dot{s}$:

$$f(t) = \varphi(\dot{s}(t)), \tag{2.17}$$

where φ is a function determined by material 2. It is assumed that $\varphi(\dot{s})$ is a continuous function that increases monotonically with $\dot{s}$. The entropy inequality (2.15) imposes the restriction

$$\varphi(\dot{s})\,\dot{s} \geq 0 \tag{2.18}$$

on the kinetic response function φ; this and the continuity of φ imply in particular that $\varphi(0)=0$.

Again following the arguments in [2, 3], we impose a nucleation criterion for the *initiation* of a phase transformation from the low-strain phase to the high-strain phase in material 2: *such a transformation takes place through the emergence at* x = 0 *of a phase boundary whenever the associated driving traction* f *would be at least as great as a given critical value* f_* *that is also determined by the nature of material* 2. After entering the bar, the phase boundary moves to the right in accordance with the kinetic relation (2.17). We assume that the critical value of driving traction satisfies

$$0 \leq f_* \leq \frac{\gamma_0}{2}(\sigma_M - \sigma_m). \tag{2.19}$$

The lower bound in (2.19) is a necessary consequence of the entropy inequality (2.15); the right inequality guarantees that material 2 will support *slowly* propagating phase boundaries as well as the fast ones permitted here; see the discussion of quasi-static phase transitions in [1-3]. Imposing the upper bound on f_* in (2.19) also simplifies the details of nucleation in the problem to be treated here, so we adopt it even though it is not strictly necessary to do so.

Wave propagation properties of the material 1/material 2 bar modeled above are ultimately to be compared with corresponding properties of a material 1/material 3 bar, in which material 3 is a linearly elastic material whose density and Young's modulus coincide with their counterparts in the *low-strain* phase of material 2: $\rho_3 = \rho_2$, $\mu_3 = \mu_2$. Thus to treat the material 1/ material 3 bar, one must modify the basic field equations (2.1)-(2.6) and jump conditions (2.8), (2.9) only to the extent of replacing $\hat{\sigma}_2(\gamma)$ by $\hat{\sigma}_3(\gamma)$ in (2.4), and thereupon replacing (2.6) by

$$\hat{\sigma}_3(\gamma) = \mu_2\,\gamma, \quad -1 < \gamma < \infty. \tag{2.20}$$

Equations (2.10)-(2.15) pertaining to the energetics of the bar remain valid but trivial, since for discontinuities in either material 1 or material 3, (2.14) always yields $f = 0$ in place of (2.16). Since phase transitions cannot occur in either material 1 or material 3, the kinetic relation relation (2.17), the attendant restriction (2.18) and the nucleation criterion all become irrelevant and are discarded.

3. The wave propagation problem. We now formulate the wave propagation problem to be considered. We suppose that, at time $t = -\infty$, an incident wave bearing a given tensile strain $\gamma_I > 0$ and a given particle velocity v_I is initiated at $x = -\infty$ in material 1, traveling to the right and striking the bimaterial interface $x = 0$ at time $t = 0-$. At $t = 0+$, a sound wave will be

reflected from the interface back into material 1, and a second sound wave will be transmitted into the right half of the bar, which heretofore was at rest and unstrained, and therefore in the low-strain phase of material 2. If the incident wave fails to be strong enough to nucleate a phase transition in material 2, only these two sound waves will be generated. On the other hand, if nucleation *does* occur, there will also be a phase boundary that emerges from the interface $x = 0$ and travels to the right, leaving the particles of material 2 that are behind it in the high-strain phase.

Thus we are given an incident wave of the form

$$\gamma(x, t) = \begin{cases} \gamma_I, & x < c_1 t, \\ 0, & x > c_1 t, \end{cases} \qquad v(x, t) = \begin{cases} v_I, & x < c_1 t, \\ 0, & x > c_1 t, \end{cases} \qquad \text{for } t < 0, \tag{3.1}$$

where γ_I and v_I are given constants, and c_1 is the sound speed in material 1. Observe that, because γ and v are piecewise constant, the field equations (2.1) - (2.5) are trivially satisfied away from $x = c_1 t$ for $t < 0$. When one applies the jump conditions (2.8), (2.9), specialized appropriately for material 1, to the discontinuity at $x = c_1 t$ for $t < 0$, one finds that

$$c_1 = (\mu_1/\rho_1)^{1/2}, \qquad v_I = - c_1 \gamma_I, \tag{3.2}$$

thus determining the sound speed in material 1 and imposing a restriction on the strain and particle velocity in the incident wave.

Given the incident wave, and assuming first that nucleation *does* occur, we seek functions $\gamma(x, t)$ and $v(x, t)$ of the following form on the upper half of the x, t-plane (see Figure 2):

$$\gamma(x,t)=\begin{cases}\gamma_I, & x<-c_1t,\\ \gamma_R, & -c_1t<x<0,\\ \bar{\gamma}_T, & 0<x<\dot{s}t,\\ \overset{+}{\gamma}_T, & \dot{s}t<x<c_2t,\\ 0, & c_2t<x<\infty,\end{cases}\qquad v(x,t)=\begin{cases}v_I, & x<-c_1t,\\ v_R, & -c_1t<x<0,\\ \bar{v}_T, & 0<x<\dot{s}t,\\ \overset{+}{v}_T, & \dot{s}t<x<c_2t,\\ 0, & c_2t<x<\infty,\end{cases}\quad \text{for } t>0. \tag{3.3}$$

here the two constants γ_R, v_R associated with the reflected wave and the four constants $\overset{\pm}{\gamma}_T$, $\overset{\pm}{v}_T$ of the transmitted wave are to be determined, as are the respective constant speeds $\dot{s}$ and c_2 of the phase boundary and the sound wave in the low-strain phase of material 2. It is assumed in (3.3) that $\dot{s}$ is less than c_2, so that the phase boundary moves subsonically in material 2; one can show that this is in fact necessary.

Since γ and v of (3.3) are piecewise constant, the field equations (2.1)-(2.6) are trivially satisfied away from discontinuities. We shall speak of the problem of determining the unknown constants in (3.3) from the jump conditions (2.8), (2.9) and the entropy inequality (2.15) at the two sound waves, the bimaterial interface and the phase bound ry as the *wave propagation problem*. We shall find that this problem has a 1-parameter family of solutions, parameter $\dot{s}$; $\dot{s}$ is then determined by enforcing the kinetic relation at the phase boundary $x = \dot{s}t$.

If the incident wave *fails* to nucleate a phase transition in material 2, the appropriate form for the solution corresponding to the incident wave (3.1) is that obtained by deleting from (3.3) the wedge $0 < x < \dot{s}t$ associated with the transformed material:

$$\gamma(x,t)=\begin{cases}\gamma_I, & -\infty<x<-c_1t,\\ \gamma_R, & -c_1t<x<0,\\ \gamma_T, & 0<x<c_2t,\\ 0, & c_2t<x<\infty,\end{cases}\qquad v(x,t)=\begin{cases}v_I, & -\infty<x<-c_1t,\\ v_R, & -c_1t<x<0,\\ v_T, & 0<x<c_2t,\\ 0, & c_2t<x<\infty,\end{cases}\quad \text{for } t>0. \tag{3.4}$$

The form (3.4) is also appropriate for the composite bar composed of materials 1 and 3. Formally, one can infer properties of the solution whose form is (3.4) from that of the form (3.3) by setting $\dot{s} = 0$ prior to enforcing the kinetic relation, thus avoiding the need to consider (3.4) separately from (3.3).

4. Determination of γ and v.

Case 1. Nucleation occurs. We first assume that the incident wave causes a phase transition to occur in material 2, and we construct the corresponding reflected and transmitted waves.

When one imposes the jump conditions (2.8), (2.9) at the reflected sound wave $x = -c_1 t$ in material 1, at the bimaterial interface x = 0, at the phase boundary $x = \dot{s}t$ and at the transmitted sound wave $x = c_2 t$ in material 2, one obtains a system of eight algebraic equations for the eight quantities γ_R, v_R, $\overset{\pm}{\gamma}_T$, $\overset{\pm}{v}_T$, c_2 and $\dot{s}$. Of these equations, the two arising from the reflected sound wave are not independent because of $(3.2)_1$, and the two arising from the transmitted sound wave determine c_2 as

$$c_2 = (\mu_2/\rho)^{1/2}, \tag{4.1}$$

and then reduce to a redundant pair. This leaves six independent equations for the seven unknowns γ_R, v_R, $\overset{\pm}{\gamma}_T$, $\overset{\pm}{v}_T$ and $\dot{s}$. These equations are readily solved for the γ's and v's in terms of $\dot{s}$, furnishing

$$\gamma_R = \frac{2}{1+\beta}\,\gamma_I - \frac{1}{\alpha(1+\beta)}\,\frac{\dot{s}}{c_2+\dot{s}}\,\gamma_0, \tag{4.2}$$

$$\bar{\gamma}_T = \frac{2\alpha\beta}{1+\beta}\,\gamma_I + \frac{\dot{s} + (1+\beta)c_2}{(1+\beta)(c_2+\dot{s})}\,\gamma_0\,, \tag{4.3}$$

$$\overset{+}{\gamma}_T = \frac{2\alpha\beta}{1+\beta}\,\gamma_I - \frac{\dot{s}\,(\beta c_2 + \dot{s})}{(1+\beta)(c_2^2 - \dot{s}^2)}\,\gamma_0\,, \tag{4.4}$$

$$v_R = \bar{v}_T = -\frac{2\,\beta}{1+\beta}\,c_1\gamma_I - \frac{1}{\alpha(1+\beta)}\,\frac{\dot{s}}{c_2+\dot{s}}\,c_1\gamma_0\,, \tag{4.5}$$

$$\overset{+}{v}_T = -\frac{2\,\beta}{1+\beta}\,c_1\gamma_I + \frac{(\beta c_2 + \dot{s})\dot{s}}{(1+\beta)(c_2^2 - \dot{s}^2)}\,c_2\gamma_0\,, \tag{4.6}$$

where we have introduced the symbols

$$\alpha = c_1/c_2\,, \qquad \beta = \rho_1 c_1/\rho_2 c_2\,, \tag{4.7}$$

for the respective ratios of the sound speeds and the mechanical impedances of materials 1 and 2, and we have also eliminated the particle velocity v_I of the incident wave from the results by using $(3.2)_2$.

The representations (4.2)-(4.6) for the strains and particle velocities involve the as yet unknown value of the phase boundary speed $\dot{s}$, which will ultimately be determined by the kinetic relation.

To assure that $\bar{\gamma}_T$ and $\overset{+}{\gamma}_T$ are respectively in the high- and low-strain phases of material 2, we must necessarily enforce the *phase segregation* inequalities $\bar{\gamma}_T \geq \gamma_m$, $\overset{+}{\gamma}_T \leq \gamma_M$. By (4.3) and (4.4), these inequalities are equivalent to

$$\gamma_I \geq G_m(\dot{s}) \equiv \frac{1+\beta}{2\alpha\beta}\,\gamma_m - \frac{1}{2\alpha\beta}\,\frac{(1+\beta)c_2 + \dot{s}}{c_2+\dot{s}}\,\gamma_0\,, \qquad 0 < \dot{s} < c_2\,, \tag{4.8}$$

$$\gamma_I \leq G_M(\dot{s}) \equiv \frac{1+\beta}{2\alpha\beta}\,\gamma_M + \frac{1}{2\alpha\beta}\,\frac{\dot{s}\,(\beta c_2 + \dot{s})}{c_2^2 - \dot{s}^2}\,\gamma_0, \qquad 0 < \dot{s} < c_2. \tag{4.9}$$

The functions G_M and G_m defined in (4.8), (4.9) are both monotonically increasing with $\dot{s}$; moreover, $G_M(0) > 0$, and $G_M(\dot{s})$ tends to $+\infty$ as $\dot{s}$ tends to c_2, while $G_m(c_2)$ is finite. The inequalities in (4.8), (4.9) are restrictions on the datum γ_I and the phase boundary speed $\dot{s}$. These restrictions are illustrated in the $\dot{s}, \gamma_I$-plane of Figure 3, where the curves Γ_m: $\gamma_I = G_m(\dot{s})$ and Γ_M: $\gamma_I = G_M(\dot{s})$ are shown schematically; in the figure, it has been assumed for definiteness that $G_m(0) > 0$ (or equivalently $\sigma_m > 0$), though this need not be the case. Only the pairs $(\dot{s}, \gamma_I)$ that correspond to points on or between these two curves are permitted by the phase segregation requirement.

To find the driving traction f at the phase boundary, one substitutes for $\bar{\gamma}_T$ and $\overset{+}{\gamma}_T$ from (4.3), (4.4) into (2.16); the result is

$$f = \frac{1}{2}\mu_2\,\gamma_0 \left\{ \frac{4\alpha\beta}{1+\beta}\,\gamma_I + \frac{\left[(1+\beta)c_2^2 - 2\beta c_2 \dot{s} - 2\dot{s}^2\right]}{(1+\beta)(c_2^2 - \dot{s}^2)}\,\gamma_0 - \gamma_M - \gamma_m \right\}. \tag{4.10}$$

The curve Γ_0 in the $\dot{s}, \gamma_I$-plane along which $f = 0$ is therefore given by

$$\Gamma_0: \quad \gamma_I = G_0(\dot{s}) \equiv \frac{1}{2}\left[G_m(\dot{s}) + G_M(\dot{s})\right]; \tag{4.11}$$

we call Γ_0 the *Maxwell curve*. For $0 < \dot{s} < c_2$, the right side of (4.11) is a monotonically increasing function of $\dot{s}$ that tends to $+\infty$ as $\dot{s}$ tends to c_2. Also, from (4.11), (4.8), (4.9), (2.6) and the assumption that the Maxwell stress σ_0 is positive, one has

$$G_0(0) = \frac{1+\beta}{4\alpha\beta}(\gamma_M + \gamma_m - \gamma_0) = \frac{1+\beta}{2\alpha\beta}\frac{\sigma_0}{\mu_2} > 0. \tag{4.12}$$

The curve Γ_0 is also shown schematically in Figure 3 . The requirement $f \geq 0$, imposed by the entropy inequality (2.15), holds only for points $(\dot{s}, \gamma_I)$ in the closed curvilinear strip between Γ_0 and Γ_M.

Finally, we must assure that $\gamma(x, t) > -1$ everywhere in the x,t-plane. One can show that, if $(\dot{s}, \gamma_I)$ lies in the strip between Γ_0 and Γ_M, the strains γ_R and $\bar{\gamma}_T$ defined in (4.2), (4.3) automatically fulfill this requirement. On the other hand, it turns out that $\overset{+}{\gamma}_T > -1$ if and only if $(\dot{s}, \gamma_I)$ lies above the curve Γ_{-1} defined by

$$\Gamma_{-1}:\ G_{-1}(\dot{s}) = -\frac{1+\beta}{2\alpha\beta} + \frac{1}{2\alpha\beta}\frac{\dot{s}\,(\beta c_2 + \dot{s})}{c_2^2 - \dot{s}^2}\gamma_0\ , \quad 0 < \dot{s} < c_2 . \tag{4.13}$$

From (4.13) and (4.9), it can be seen that Γ_M always lies above Γ_{-1}. After some algebra, one shows that Γ_{-1} intersects Γ_0 exactly once; the intersection occurs at a value $\dot{s} = \dot{s}_*$ given by

$$\dot{s}_* = \left(1 - \frac{\gamma_0}{\gamma_M + \gamma_m + 2}\right)^{1/2} c_2\ <\ c_2 . \tag{4.14}$$

For values of $\dot{s}$ to the left of $\dot{s}_*$, Γ_{-1} lies below Γ_0, while for $\dot{s}_* < \dot{s} < c_2$, the reverse is true.

Let S stand for the closed curvilinear strip bounded above by Γ_M and below in part by Γ_0 and in part by Γ_{-1}; S is shown hatched in Figure 3. It then follows that *for each value of the*

incident strain γ_I *that lies in the interval* $(G_0(0), +\infty)$, *there is a 1-parameter family* (*parameter* $\dot{s}$) *of admissible fields* $\gamma(x, t)$, $v(x, t)$ *of the form (4.2)-(4.6) that fulfill all of the conditions of the wave propagation problem.* For each given γ_I in this interval, the permissible range of the parameter $\dot{s}$ is that corresponding to the end-points of the associated horizontal line segment in the $\dot{s}$, γ_I-plane connecting either Γ_M or the vertical axis on the left to either Γ_0 or Γ_{-1} on the right, as appropriate. Each of these fields that corresponds to a positive value of $\dot{s}$ involves a phase transition. For each value of γ_I *outside* $(G_0(0), +\infty)$, there is *no* solution to the wave propagation problem that involves a phase transition; as we shall see in Case 2 below, the only solution available for such an initial datum γ_I is one in which no phase transition occurs.

Case 2. Nucleation does not occur. In this case, we can find the solutions of the form (3.4) arising from the incident wave (3.1) by formally setting $\dot{s} = 0$ in the expressions (4.2)-(4.6). This yields

$$\gamma(x,t) = \begin{cases} \dfrac{2}{1+\beta}\,\gamma_I, & -c_1 t < x < 0, \\ \dfrac{2\alpha\beta}{1+\beta}\,\gamma_I, & 0 < x < c_2 t, \end{cases} \qquad v(x,t) = -\frac{2\alpha\beta}{1+\beta}\,c_2\gamma_I, \quad -c_1 t < x < c_2 t, \qquad t > 0, \tag{4.15}$$

for the reflected and transmitted waves in the absence of the phase transformation. Since the solution (4.15) involves only sound waves, the entropy inequality (2.15) holds automatically. The phase segregation requirement demands that $\gamma(x, t) \le \gamma_M$ for $0 < x < c_2 t$; from (4.15) and (4.9), this leads to

$$\gamma_I \le G_M(0). \tag{4.16}$$

For each positive initial datum γ_I satisfying (4.16), the fields (4.15) comprise a solution of the

wave propagation problem that is uniquely determined by the data and in which no phase transition takes place. Since the incident strain γ_I has been assumed positive (tensile) throughout, the requirement $\gamma(x, t) > -1$ is automatically satisfied by the strain field of $(4.15)_1$.

It is important to note that (4.15) *also* represents the solution to the wave propagation problem in which the left half of the bar is composed of material 1, while the right half is made of material 3. In this case, however, (4.15) represents a solution for *all* tensile values of the initial datum γ_I, and not merely for those satisfying (4.16).

Combining the conclusions reached in Cases 1 and 2 establishes the following results.
(i) For each value of the initial datum γ_I in the interval $(0, G_0(0))$, there is exactly one solution to the wave propagation problem; it is given by (4.15) and does not involve a phase transition. (ii) For each γ_I in $(G_0(0), G_M(0)]$, there is a 1-parameter family of solutions (4.2)-(4.6) to the wave propagation problem that involve a phase transition, *and* a single solution (4.15) that does not. (iii) For values of $\gamma_I > G_M(0)$, there is only the 1-parameter family of soutions (4.2)-(4.6), each of which involves a phase transition.

The ambiguity remaining when γ_I is in $(G_0(0), G_M(0)]$ (case (ii) above) is resolved by first invoking the nucleation criterion, then the kinetic relation. Setting f in (4.10) equal to the nucleation value f_* defines a curve Γ_*: $\gamma_I = G_*(\dot{s})$ in the $\dot{s}$, γ_I-plane; we omit the formula for G_*, and we do not show Γ_* in Figure 3. The constitutive inequality (2.19) can be shown to guarantee that Γ_* always lies in the between Γ_0 and Γ_M, and that $G_0(0) \le G_*(0) \le G_M(0)$. We assume further that f_* is such that Γ_* lies in the slightly smaller strip *S* in Figure 3. At points on or between Γ_* and Γ_M, (4.10) yields values of f at least as great as f_*. It then follows that nucleation will occur if the incident strain γ_I is at least as great as the critical value γ_I^* given by

$$\gamma_I^* = G_*(0) = \frac{1+\beta}{2\alpha\beta\mu_2\gamma_0}(\sigma_0\gamma_0 + f_*) \ . \tag{4.17}$$

When the incident wave carries a strain γ_I that is less than γ_I^*, no phase transition occurs, and the appropriate solution to the wave propagation problem is that given by (4.15). On the other hand, if γ_I is at least as great as γ_I^*, nucleation will take place in material 2, a phase boundary will emerge at x = 0, and the appropriate solution must be selected from the 1-parameter family of admissible fields (4.2)-(4.6) involving a phase transition. The kinetic relation (2.17) provides the mechanism for this selection. Substituting for f from (4.10) into (2.17) yields the equation for $\dot{s}$:

$$\gamma_I = G_k(\dot{s}) \equiv G_0(\dot{s}) + \frac{1}{\mu_2\gamma_0}\frac{1+\beta}{2\alpha\beta}\varphi(\dot{s}) \ , \quad \gamma_I \geq \gamma_I^* \ , \tag{4.18}$$

where φ is the kinetic response function of the material. Because $\varphi(\dot{s})$ and $G_0(\dot{s})$ both increase monotonically with $\dot{s}$, the same is true of the right side of (4.18). Since $\varphi(0) = 0$, one has $G_k(0) = G_0(0)$; moreover, $G_k(\dot{s})$ tends to $+\infty$ as $\dot{s}$ tends to c_2, and $0 < G_0(\dot{s}) < G_k(\dot{s}) < G_M(\dot{s})$ for $0 < \dot{s} < c_2$. Thus the curve Γ_k in the $\dot{s}$, γ_I-plane represented by (4.18) lies between Γ_0 and Γ_M. We further assume that φ is such that Γ_k lies in the strip *S* (see Figure 3). Clearly (4.18) determines exactly one value of $\dot{s}$ for each given γ_I in $(G_0(0), +\infty)$. When inserted into (4.2)-(4.6), this value of $\dot{s}$ completes the determination of the response to the incident wave when a phase transition is nucleated.

5. Reflectivity. For a given incident wave in material 1, we wish to compare the strengths of the reflected waves when (i) the right half of the bar consists of material 2, and (ii) the right half is composed of material 3. To motivate our notion of the *strength* of the transmitted or reflected wave, it is helpful to reconsider the energetics of the composite bar. Consider first the case of the material 1/material 2 bar. By extending the energy considerations

of Section 2 to a portion of the bar long enough to include the four discontinuities at $x = -c_1 t$, $x = 0$, $x = \dot{s}t$ and $x = c_2 t$, one can establish the following identity among energy rates for fields of the form (3.3):

$$e_I = e_R + e_T + fA\dot{s}\,, \tag{5.1}$$

where

$$e_I = \left\{ \frac{1}{2}\, c_1^2 \gamma_I^2 + \frac{1}{2} v_I^2 \right\} \rho_1 c_1 A\,, \tag{5.2}$$

$$e_R = \left\{ \frac{1}{2} c_1^2 (\gamma_R - \gamma_I)^2 + \frac{1}{2} (v_R - v_I)^2 \right\} \rho_1 c_1 A\,, \tag{5.3}$$

$$e_T = \left\{ \frac{W_2(\bar{\gamma}_T)}{\rho_2} + \frac{1}{2} \bar{v}_T^2 \right\} \rho_2 \dot{s} A + \left\{ \frac{1}{2} c_2^2 \overset{+}{\gamma}{}_T^2 + \frac{1}{2} \overset{+}{v}{}_T^2 \right\} \rho_2 (c_2 - \dot{s}) A, \tag{5.4}$$

f is the driving traction at the phase boundary, A is the cross-sectional area of the bar and $\dot{s}$ is the phase boundary speed. In (5.3), (5.4), γ_R, $\overset{\pm}{\gamma}_T$, v_R and $\overset{\pm}{v}_T$ are given in terms of $\dot{s}$ by (4.2)-(4.6), and $W_2(\bar{\gamma}_T)$ refers to the stored energy density W_2 of material 2 . We call e_I, e_R and e_T the incident, reflected and transmitted energy rates; note that all have the units of energy per unit time, and all are non-negative. By (2.15), the dissipation rate $fA\dot{s}$ is non-negative as well. Thus neither e_R nor e_T is greater than e_I .

Setting $\dot{s} = 0$ in (5.1)-(5.4) provides the corresponding identity among energy rates for the material 1/material 3 bar in which no phase transformation can occur.

The reflectivity and transmissivity are defined by

$$q_R = \frac{e_R}{e_I} \le 1\,, \qquad q_T = \frac{e_T}{e_I} \le 1, \tag{5.5}$$

respectively; they are measures of the strengths of the reflected and transmitted waves relative to the incident wave. We are interested in

$$Q_R(\dot{s}, \gamma_I) = \frac{q_R}{q_R\big|_{\dot{s}=0}}\,, \qquad Q_T(\dot{s}, \gamma_I) = \frac{q_T}{q_T\big|_{\dot{s}=0}}\,, \tag{5.6}$$

which are the ratios of reflectivity and transmissivitiy in the material 1/material 2 bar to their respective counterparts in the material 1/material 3 composite. We study Q_R in the present section, Q_T in the next. When $\beta = 1$, the reflectivity $q_R|_{\dot{s}=0}$ of the material 1/material 3 bar vanishes, and $(5.6)_1$ fails to define Q_R; we exclude this possibility for simplicity by assuming that $\beta \neq 1$.

The reflectivity ratio Q_R is a function on the admissible strip *S* shown hatched in Figure 3. For a given kinetic relation, only the values of Q_R at points on the associated kinetic curve Γ_k are relevant; for a given value of the strain γ_I in the incident wave, only the value of Q_R at the point on Γ_k whose ordinate is γ_I is relevant. If, for example, one has $Q_R(\dot{s}, \gamma_I) < 1$ at this point, then the reflectivity of the material 1/material 2 bar is less than that of the material 1/material 3 bar, so that reflection has been diminished by the occurrence of the phase transition. We study the properties of Q_R as a function on *S*.

From $(5.6)_1$, $(5.5)_1$, (5.3), (4.2) and (4.5), one finds

$$Q_R(\dot{s}, \gamma_I) = \left\{ 1 - \frac{1}{\alpha(1-\beta)} \frac{\gamma_0}{\gamma_I} \frac{\dot{s}}{c_2 + \dot{s}} \right\}^2 ; \qquad (5.7)$$

it may be noted from Figure 3 that $\gamma_I \geq G_0(0) > 0$ at all points on S; this fact, together with the assumption that $\beta \neq 1$, guarantees that Q_R is well defined on S. Of course, $Q_R(0, \gamma_I) = 1$; we now determine the locus of points $(\dot{s}, \gamma_I)$ in the $\dot{s}, \gamma_I$-plane at which $Q_R = 1$ and $\dot{s} \neq 0$; such points correspond to situations in which the reflectivity is unaltered by the phase transition. This locus consists of the interior points on the curve Γ_R^1 defined by

$$\Gamma_R^1: \quad \gamma_I = \frac{1}{2\alpha(1-\beta)} \frac{\dot{s}}{c_2 + \dot{s}} \gamma_0, \quad 0 \leq \dot{s} \leq c_2 . \qquad (5.8)$$

Only those points on Γ_R^1, if any, that lie in the admissible strip S are relevant. If a portion of Γ_R^1 lies in the interior of S, the strip is divided into two parts: on one, $Q_R < 1$, while on the other, $Q_R > 1$. Whether Γ_R^1 has points in common with the interior of S is determined by two material parameters β and λ, where $\beta = \rho_1 c_1 / \rho_2 c_2$ is the (positive) impedance ratio introduced earlier, and λ is defined by

$$\lambda = \frac{\gamma_M + \gamma_m}{\gamma_0} . \qquad (5.9)$$

By (2.7) and our assumption that the Maxwell stress σ_0 is positive, we have $\lambda > 1$. We first show that Γ_R^1 *fails* to intersect the interior of S if *either* of the following conditions holds:

$$\text{(i) } \beta > 1 \ \ \textit{or} \ \ \text{(ii) } 0 < \beta < 1 \text{ and } \lambda > \Lambda(\beta^2), \qquad (5.10)$$

where

$$\Lambda(z) = 1 + \frac{1}{2}\left\{ \frac{1 + z - [(1 - z)(1 + 3z)]^{1/2}}{1 - z} \right\}, \quad 0 < z < 1. \tag{5.11}$$

One can show that $\Lambda(z)$ increases monotonically from the value 1 at $z = 0$, tending to $+\infty$ as z tends to 1 .

To establish (5.10), we first note that by (4.11), (4.12) and (5.8), the curve Γ_R^1 lies below the Maxwell curve Γ_0 near both endpoints of the interval $0 < \dot{s} < c_2$, so that Γ_R^1 cannot have points in common with the interior of the admissible strip S unless Γ_R^1 intersects Γ_0 at least twice. To investigate such intersections, one equates the right sides of (4.11) and (5.8), obtaining a quadratic equation for $\dot{s}$. It is readily shown that (5.10) describes precisely the conditons under which this equation has no real roots in the interval $(0, c_2)$. Thus (5.10) is indeed *sufficient* to assure that Γ_R^1 fails to intersect the interior of S. Conversely, when

$$\beta < 1 \text{ and } \lambda < \Lambda(\beta^2), \tag{5.12}$$

the quadratic equation mentioned above has two real roots in $(0, c_2)$, corresponding to two points at which Γ_R^1 and Γ_0 intersect. By a calculation too lengthy to be included here, one can show that the smaller of these two roots lies in $(0, \dot{s}_*)$ (Figure 3), so that at the left point of intersection, Γ_R^1 indeed enters S.

We are now in a position to delineate conditions under which the occurrence of the phase transition always increases the reflectivity, always decreases it or might do either. Figure 4 shows three open regions marked I, II and III in the quadrant $\beta > 0$, $\lambda > 1$ in the β, λ- plane.

Region III corresponds precisely to the inequalities (5.12) and hence to the case in which Γ_R^1 enters S. It follows that the reflectivity ratio Q_R is less than one at some points (β, λ) in this region, greater than one at others; the actual effect obtained depends on the particular kinetic relation and possibly on the particular value of the strain γ_I in the incident wave as well. The region marked I corresponds to values of β and λ for which, on the interior of the admissible strip S of Figure 3, reflectivity at the interface is always *less* in the material 1/material 2 bar than it is in the material 1/material 3 bar. Thus assuming that nucleation has occurred, the phase transformation induced in material 2 by the incident wave *always* acts to reduce reflection if (β, λ) corresponds to a point in region I, regardless of the particular kinetic relation involved . In region II, precisely the reverse is true: reflectivity is always *increased* at points in the interior of S by the phase transformation. For values of (β, λ) belonging to the various regions I, II and III, Figure 5 schematically shows the curve Γ_R^1 in the $\dot{s}$, γ_I-plane.

If the impedance of material 1 is greater than that of the parent phase of material 2, then $\beta > 1$, and it follows from (5.7) that $Q_R(\dot{s}, \gamma_I)$ increases with $\dot{s}$ for each fixed γ_I . Thus at each given γ_I, the maximum value of the reflectivity ratio Q_R occurs on the right boundary of S, while the minimum value occurs on the left boundary. Thus when $\beta > 1$, fast kinetics promote the increase of reflectivity, while slow kinetics are best for reducing it.

If the impedance ratio $\beta < 1$, the behavior of Q_R as a function of phase boundary speed $\dot{s}$ at fixed incident strain γ_I is more complicated; under certain circumstances, Q_R may vanish at some points in S, corresponding through (5.7) to a minimum in Q_R.

6. Transmissivity. We state here without proof some results pertaining to the transmissivity ratio Q_T, whose behavior is more complicated than that of the reflectivity ratio Q_R. Using $(5.6)_2$, $(5.5)_2$, (5.4), and (4.3) - (4.6), one can show that

$$Q_T(\dot{s}, \gamma_I) = 1 - \frac{1}{2\alpha\beta} \frac{\gamma_0}{\gamma_I} T_1(\dot{s}) + \frac{1}{8\alpha^2\beta^2} \left(\frac{\gamma_0}{\gamma_I}\right)^2 T_2(\dot{s}), \tag{6.1}$$

where

$$T_1(\dot{s}) = [2\beta + (1 + \beta)\dot{s}/c_2] \frac{\dot{s}}{c_2 + \dot{s}} , \tag{6.2}$$

$$T_2(\dot{s}) = (\lambda - 1)(1 + \beta)^2 \dot{s}/c_2 + \frac{c_2 \dot{s}^2}{(c_2^2 - \dot{s}^2)(c_2 + \dot{s})} T_3(\dot{s}), \tag{6.3}$$

$$T_3(\dot{s}) = 2\beta^2 + (1 + 4\beta + \beta^2) \frac{\dot{s}}{c_2} + (1 - \beta^2) \frac{\dot{s}^2}{c_2^2} , \tag{6.4}$$

and λ is defined in (5.9). It is possible to show that, for $\beta > 1$, one has $Q_T < 1$ at *all* interior points of the admissible strip S; on the other hand if $\beta < 1$, one finds that Q_T takes values greater than 1 at some points in S, less than 1 at others.

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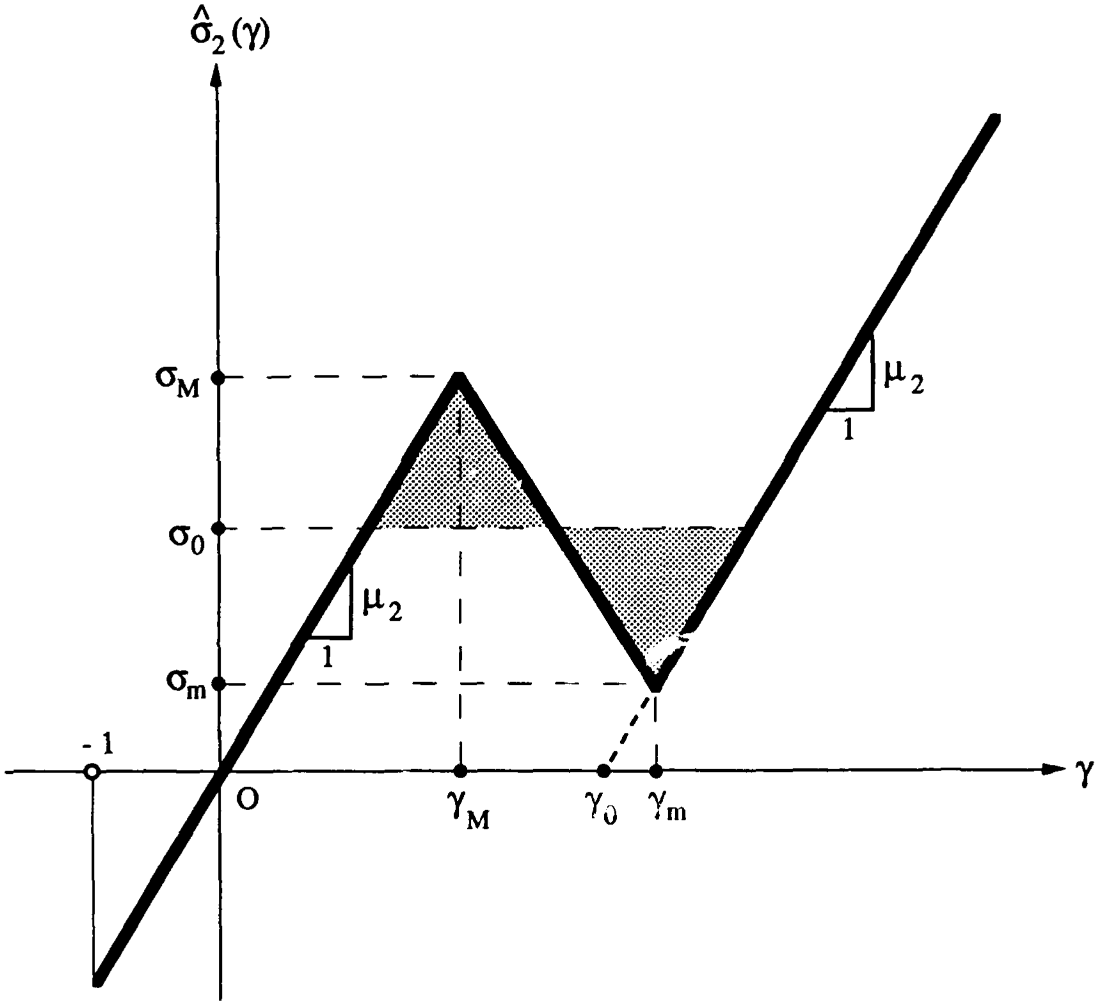

FIGURE 1. STRESS-STRAIN CURVE FOR MATERIAL 2.

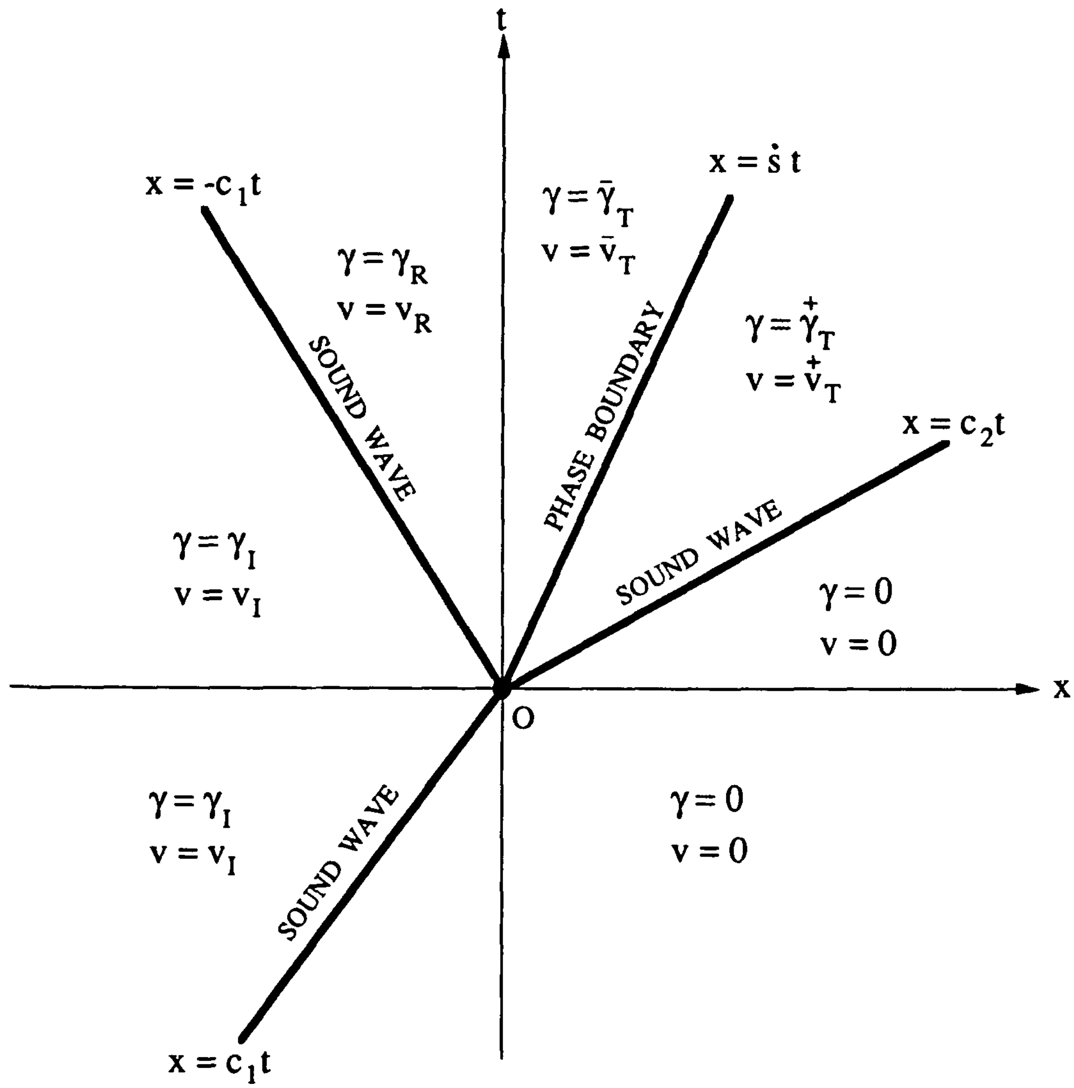

FIGURE 2. INCIDENT, REFLECTED AND TRANSMITTED WAVES.

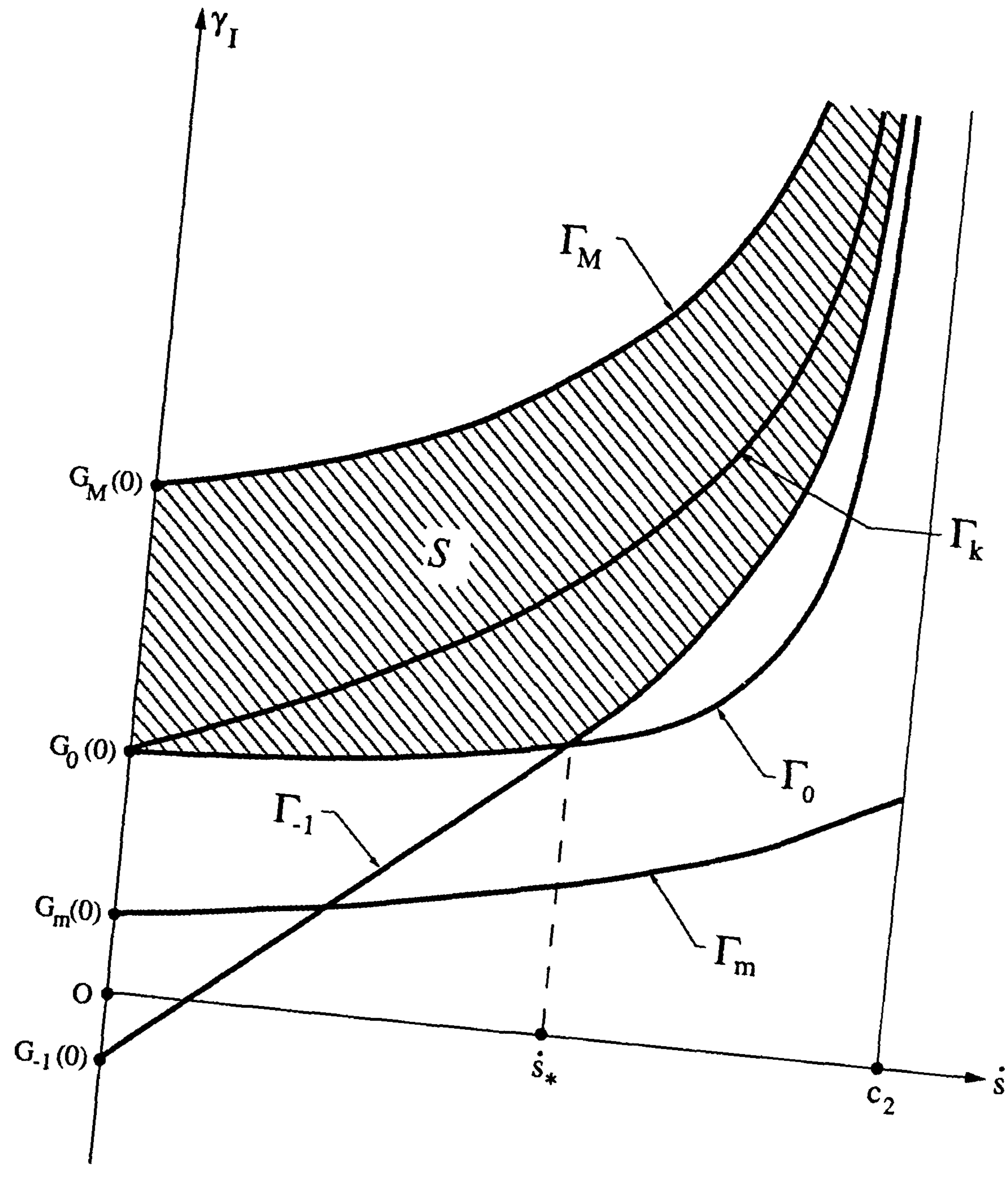

FIGURE 3. THE ADMISSIBLE STRIP S IN THE $\dot{s}$, γ_I- PLANE.

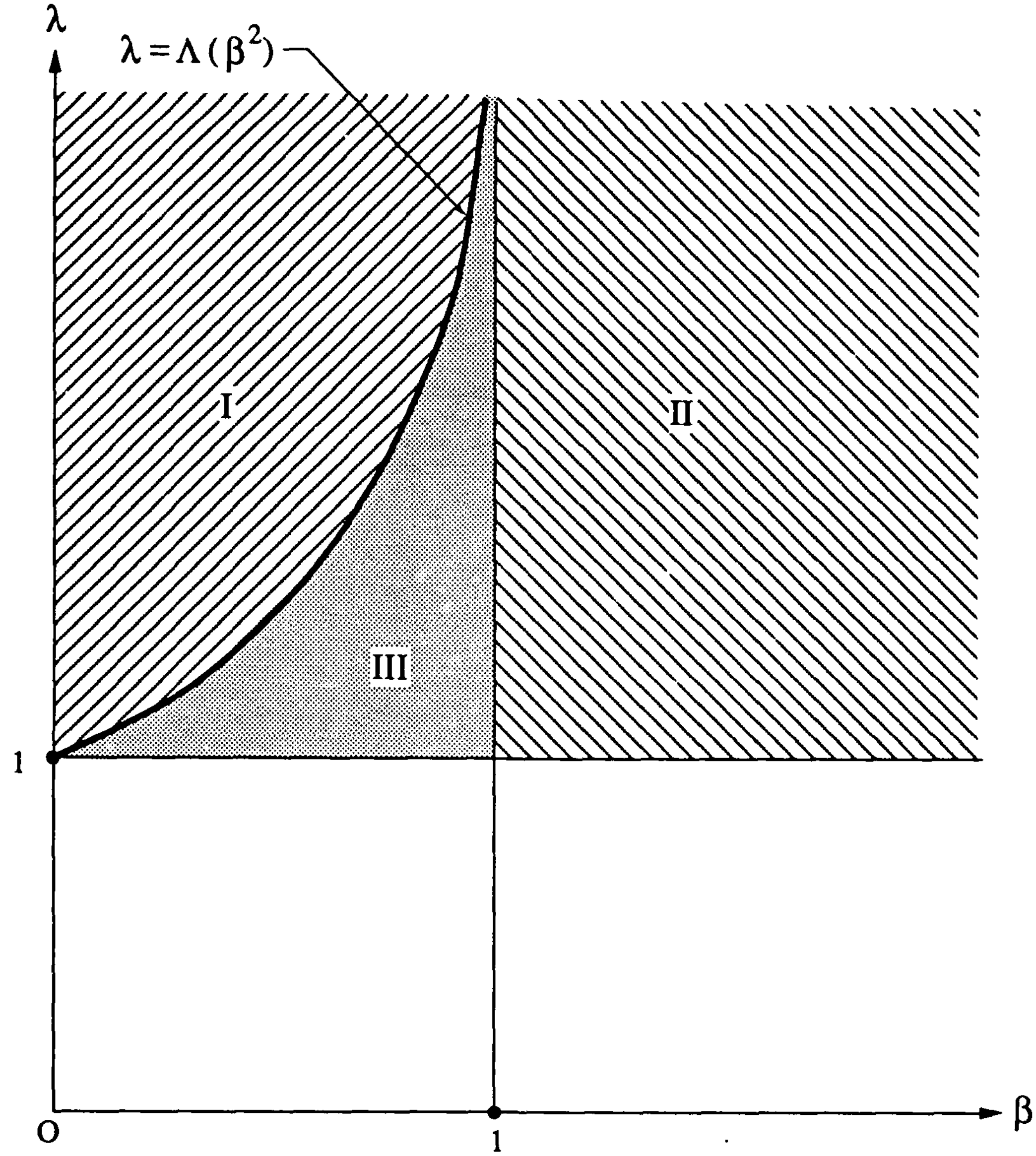

FIGURE 4. REGION I : $Q_R < 1$;
REGION II : $Q_R > 1$;
REGION III : $Q_R - 1$ MAY HAVE EITHER SIGN.

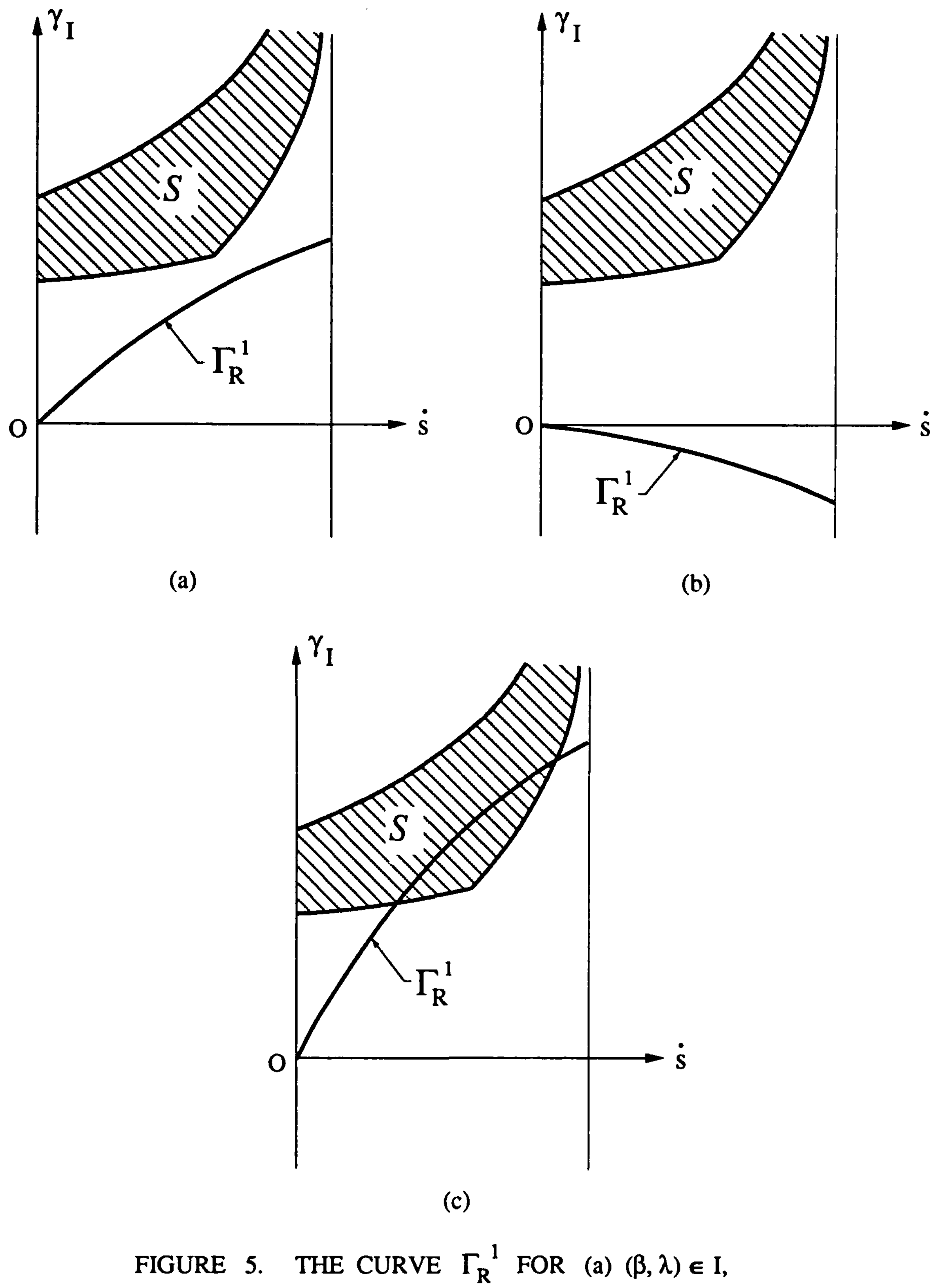

FIGURE 5. THE CURVE $\Gamma_R^{\ 1}$ FOR (a) $(\beta, \lambda) \in \mathrm{I}$, (b) $(\beta, \lambda) \in \mathrm{II}$, (c) $(\beta, \lambda) \in \mathrm{III}$.

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This paper is concerned with waves in a composite elastic bar, the left half of which is composed of a linearly elastic material, while the nonlinearly elastic material of the right half can undergo a phase transition. We assume that a wave in the left portion of the bar is incident upon the interface between the two materials, and we investigate the question of whether the phase transition can be exploited to augment or diminish the strength of the reflected or transmitted wave.

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